

Unification of Program Expressions with Recursive Bindings

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Unification as a core procedure for

- **automated reasoning on programs and program transformations** w.r.t. operational semantics
- for program calculi with higher-order constructs and recursive bindings, e.g.

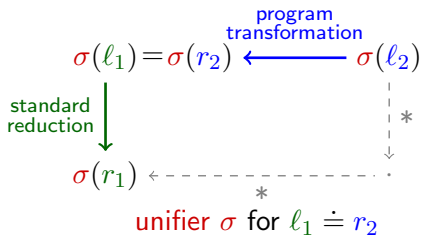
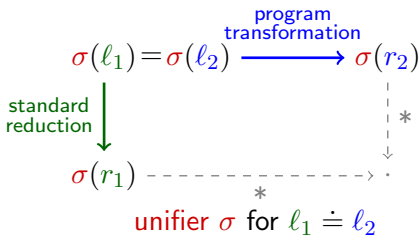
$$\text{letrec } x_1 = s_1; \dots; x_n = s_n \text{ in } t$$

- special focus: extended call-by-need lambda calculi with letrec that model core languages of **lazy functional programming languages** like Haskell

Program transformation T is **correct** iff $\forall l \rightarrow r \in T: \forall C: C[l] \downarrow \iff C[r] \downarrow$
 where $\downarrow =$ successful evaluation w.r.t. standard reduction

Diagram-based proof method to show correctness of transformations:

- Compute **overlaps** between **standard reductions** and **program transformations** (automatable by **unification**)
- Join the overlaps \Rightarrow forking and commuting diagrams
- Induction using the diagrams (automatable, see [RSSS12, IJCAR])



Operational semantics of typical call-by-need calculi (excerpt)

Reduction contexts:

$A ::= [\cdot] \mid (A e)$

$R ::= A \mid \text{letrec } Env \text{ in } A \mid \text{letrec } \{x_i = A_i[x_{i+1}]\}_{i=1}^{n-1}, x_n = A_n, Env, \text{ in } A[x_1]$

Standard-reduction rules and some program transformations

(SR,lbeta) $R[(\lambda x.e_1) e_2] \rightarrow R[\text{letrec } x = e_2 \text{ in } e_1]$

(SR,llet) $\text{letrec } Env_1 \text{ in letrec } Env_2 \text{ in } e \rightarrow \text{letrec } Env_1, Env_2 \text{ in } e$

(T,cpx) $T[\text{letrec } x = y, Env \text{ in } C[x]] \rightarrow T[\text{letrec } x = y, Env \text{ in } C[y]]$

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Meta-syntax must be capable to represent:

- contexts of different classes
- environments Env_i ,
- environment chains $\{x_i = A_i[x_{i+1}]\}_{i=1}^{n-1}$

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Variables	$x \in \mathbf{Var} ::= X$ x	(variable meta-variable) (concrete variable)
Expressions	$s \in \mathbf{Expr} ::= S$ $D[s]$ letrec env in s var x $(f r_1 \dots r_{ar(f)})$ where r_i is $o_i, s_i,$ or x_i specified by f	(expression meta-variable) (context meta-variable) (letrec-expression) (variable) (function application)
	$o \in \mathbf{HExpr}^n ::= x_1 \dots x_n . s$	(higher-order expression)
Environments	$env \in \mathbf{Env} ::= \emptyset$ $E; env$ $Ch[x, s]; env$ $x.s; env$	(empty environment) (environment meta-variable) (chain meta-variable) (binding)

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Unification-problems have to treat (implicit) restrictions on scoping and emptiness, e.g.:

- (gc): Env must not be empty; side condition on variables,
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- (cpX): x, y are not captured by C in $C[x]$

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A letrec unification problem is a tuple $P = (\Gamma, \Delta_1, \Delta_2, \Delta_3)$ with

- Γ : **unification equations** $s \doteq s'$
- Δ_1 : **non-empty contexts** (set of D -variables)
- Δ_2 : **non-empty environments** (set of E -variables)
- Δ_3 : **non-capture constraints** (set of (expression, context)-pairs)

Occurrence restrictions:

- Each S -variable occurs **at most twice** in Γ
- Each E -, Ch -, D -variable occurs **at most once** in Γ
- Ch -variables are only allowed in one letrec-environment in Γ

Unifier and Solution of $P = (\Gamma, \Delta_1, \Delta_2, \Delta_3)$

A substitution ρ is a **unifier of P** iff

- $\rho(s) \sim_{let} \rho(s')$ for all $s \doteq s' \in \Gamma$
- $\rho(D) \neq [\cdot]$ for all $D \in \Delta_1$ and $\rho(E) \neq \emptyset$ for all $E \in \Delta_2$
- $Var(\rho(s)) \cap CV(\rho(d)) = \emptyset$ for all $(s, d) \in \Delta_3$

A unifier ρ is a **solution of P** if ρ is a ground substitution.

\sim_{let} = syntactic equality upto permuting bindings in environments

$CV(d)$ = variables that are captured by the hole of context d

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Theorem (NP-Hardness)

The decision problem whether a solution for a letrec unification problem exists is NP-hard.

Proof by a reduction from MONOTONE ONE-IN-THREE-3-SAT.

Intermediate **data structure** of the algorithm: (Sol, Γ, Δ) where

- Sol is a computed substitution
- Γ is a set of equations
- $\Delta = (\Delta_1, \Delta_2, \Delta_3, \Delta_4)$
- $(\Delta_1, \Delta_2, \Delta_3)$ are constraints as in a letrec unification problem
- Δ_4 are environment equations $E_1; \dots; E_n = Ch[x, s]$

Input:

For $P = (\Gamma, \Delta_1, \Delta_2, \Delta_3)$, UnifLRS starts with $(Id, \Gamma, (\Delta_1, \Delta_2, \Delta_3, \emptyset))$

Output (on each branch):

Fail or final state (Sol, \emptyset, Δ)

Inference rules of the form $\frac{P}{P_1 \mid \dots \mid P_n}$

Four kinds of rules:

- First-order rules
- Rules for environment equations
- Rules for equations $D[s] \doteq s'$
- Failure rules

Rule application is non-deterministic:

- don't care non-determinism between the rules
- don't know non-determinism between $P_1 \mid \dots \mid P_n$

$$\frac{(Sol, \Gamma \cup \{x \doteq x\}, \Delta)}{(Sol, \Gamma, \Delta)}$$

$$\frac{(Sol, \Gamma \cup \{S \doteq s\}, \Delta)}{(Sol \circ \{S \mapsto s\}, \Gamma[s/S], \Delta[s/S])}$$
 if S is not a proper sub-expression of s

$$\frac{(Sol, \Gamma \cup \{\text{letrec } env_1 \text{ in } s_1 \doteq \text{letrec } env_2 \text{ in } s_2\}, \Delta)}{(Sol, \Gamma \cup \{env_1 \doteq env_2, s_1 \doteq s_2\}, \Delta)}$$

Unifying bindings and chains:

$$(Sol, \Gamma \cup \{x.t; env_1 \doteq Ch[y, s]; env_2\}, \Delta)$$

$$(Sol \circ \sigma, \Gamma \cup \{x.t \doteq y.D[s], env_1 \doteq env_2\}, \Delta\sigma)$$

$$\sigma = \{Ch[y, s] \mapsto y.D[s]\}$$

“equal”

$$| (Sol \circ \sigma, \Gamma \cup \{x.t \doteq y.D[\text{var } Y], env_1 \doteq Ch_2[Y, s]; env_2\}, \Delta\sigma)$$

$$\sigma = \{Ch_1[y, s] \mapsto y.D[\text{var } Y]; Ch_2[Y, s]\}$$

“prefix”

$$| (Sol \circ \sigma, \Gamma \cup \{x.t \doteq Y_1.D[\text{var } Y_2], env_1 \doteq Ch_1[y, \text{var } Y_1]; Ch_2[Y_2, s]; env_2\}, \Delta\sigma)$$

$$\sigma = \{Ch[y, s] \mapsto Ch_1[y, (\text{var } Y_1)]; Y_1.D[\text{var } Y_2]; Ch_2[Y_2, s]\}$$

“infix”

$$| (Sol \circ \sigma, \Gamma \cup \{x.t \doteq Y_1.D[s], env_1 \doteq Ch_2[y, \text{var } Y_1]; env_2, \Delta\sigma\})$$

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“suffix”

Keep chain-equations as constraints

$$\frac{(Sol, \Gamma \cup \{E_1; \dots; E_n \doteq Ch[y, s]\}, (\Delta_1, \Delta_2, \Delta_3, \Delta_4))}{(Sol, \Gamma, (\Delta_1, \Delta_2, \Delta_3, \Delta_4 \cup \{E_1; \dots; E_n \doteq Ch[y, s]\}))}$$

Standard cases:

$$\frac{(Sol, \Gamma \cup \{(x_1 \doteq x_2)\}, \Delta)}{Fail}$$

$$\frac{(Sol, \Gamma \cup \{(S \doteq s)\}, \Delta)}{Fail} \text{ if } S \text{ is a proper subterm of } s$$

Checking non-capture constraints:

$$\frac{(Sol, \Gamma, (\Delta_1, \Delta_2, \Delta_3 \cup \{(s, d)\}, \Delta_4))}{Fail} \text{ if } Var(s) \cap CV(d) \neq \emptyset$$

For a final state (Sol, \emptyset, Δ) satisfiability of Δ_4 is checked:

Guess an instantiation σ for all $E_1; \dots; E_n \doteq Ch[y, s] \in \Delta_4$ s.t.

- $\sigma(Ch[y, s]) = y.D_1[Y_1]; Y_1.D_2[Y_2]; \dots; Y_k.D_{k+1}[s]$
- $\sigma(E_i) \subseteq \{y.D_1[Y_1]; Y_1.D_2[Y_2]; \dots; Y_k.D_{k+1}[s]\}$ and $\sigma(E_i) \neq \emptyset$ if $E_i \in \Delta_2$
- $\sigma(E_1; \dots; E_n) \sim_{let} \sigma(Ch[y, s])$
- all non-capture constraints in $\Delta_3\sigma$ are valid

Deliver Fail if no such instantiation exists.

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Key Lemma

It suffices to test only those k with $k + 1 \leq M_1^2 * (M_2 + 1) + M_2$ where $M_1 = |\Delta_2 \cap \{E_1; \dots; E_n\}|$ and $M_2 = n - M_1$.

Thus, the Δ_4 -check can be done in nondeterministic polynomial time.

Proposition (Soundness)

For input P and successful output (Sol, \emptyset, Δ) :

- All ground instances of Sol that do not violate Δ are solutions of P .
- There exists at least one ground instance of Sol which solves P .

Proposition (Completeness)

For any solution ρ of a letrec unification problem P there exists a final state (Sol, \emptyset, Δ) of UnifLRS s.t. ρ is an instance of Sol .

Theorem

UnifLRS is sound and complete.

Theorem

UnifLRS **terminates in nondeterministic polynomial time** and solutions are of polynomial size.

Corollary

The letrec unification problem is NP-complete.

Implementation available from <http://goethe.link/lrsx>

- unification of expressions
- calculus descriptions as input for computing overlaps

Experiments with two call-by-need calculi:

- L_{need} : lambda calculus plus letrec
- LR: L_{need} + data constructors + case expressions + seq-expressions
- overlaps for 11 transformations w.r.t. all standard reduction rules

Statistics:

number of standard rules	Calculus L_{need}		Calculus LR	
	forking	commuting	forking	commuting
number of critical pairs	1741	2156	34319	37016
execution time (sec.)	2	3	44	56

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Further work:

- **Join the critical pairs:** Requires matching-algorithm, but also handling of the $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ -constraints, and probably some kind of meta alpha-renaming
- Equivalence of **different reductions strategies:** computing overlaps requires to unify chain-variables
($Ch_1[y, s] \doteq Ch_2[y', s']$)